

EXTREMAL VALUES IN RECURSIVE TREES VIA A NEW TREE GROWTH PROCESS

LAURA ESLAVA

ABSTRACT. We give convergence rates on the number of vertices with degree at least $c \ln n$, $c \in (1, 2)$, in random recursive trees on n vertices. This allows us to extend the range for which the distribution of the number of vertices of a given degree is well understood.

Conceptually, the key innovation of our work lies in a new tree growth process $((T_n, \sigma_n), n \geq 1)$ where T_n is a rooted labeled tree on n vertices and σ_n is a permutation of the vertex labels. The shape of T_n has the same law as that of a random recursive tree. Interesting on its own right, this process obtains T_n from T_{n-1} by a procedure we call Robin-Hood pruning, which attaches a vertex labeled n to T_{n-1} and rewires some of the edges in T_{n-1} towards the newly added vertex. Additionally, $((T_n, \sigma_n), n \geq 1)$ can be understood as a new coupling of all finite Kingman's coalescents.

1. INTRODUCTION

In a paper of 1970 [16], Na and Rapoport presented the problem of modeling how the structure of networks (as sociograms, communication and acquaintance networks) emerge through time. They considered two cases. First, a class of ‘growing’ trees whose construction corresponds to the standard construction of random recursive trees (RRTs). These are constructed by sequentially adding new vertices, which are attached to a uniformly random vertex in the previous tree. Second, a class of ‘static’ trees formed via a coalescent process beginning with n isolated nodes. They described the construction of a ‘static’ tree with n vertices as follows.

“Initially, single elements move about at random. Each collision forms a couple. A collision of a couple with a single element forms a triple, a collision of an s -tuple with a t -tuple forms an $(s + t)$ -tuple, and so on. At each collision a link is established between an element of one X -tuple and an element of another, the links being rigid so that the elements of the same k -tuple cannot collide. The process goes on until the entire set of n elements has been joined into an n -tuple.”

The term ‘static’ was motivated by the fact that this construction starts with the n vertices the tree is aimed to have at the end of the process. This is a description, in fact, of a discrete multiplicative coalescent which is linked to Kruskal’s algorithm for the minimum weighted spanning tree problem [1]¹. A growing process of such coalescent was not foreseen; however, it is now known that, for some coalescent procedures (e.g. additive

Date: January 6th 2017.

2010 Mathematics Subject Classification. 60C05, 05C80.

Key words and phrases. Kingman’s coalescent, random recursive trees, extreme values, tree growth processes, Chen-Stein method, coupling.

¹Unfortunately, it was incorrectly presumed in [16] to build uniformly random unrooted labeled trees.

and Kingman's), the resulting tree can also be constructed by a growth process [1, 15, 17]. In particular, Kingman's coalescents correspond to RRTs; see e.g. [1], or Propositions 1.1 and 5.3 below.

The key conceptual contribution of this work is what we call the Robin-Hood pruning procedure. This is a random construction which, given a Kingman's coalescent on n vertices, produces a Kingman's coalescent on $n + 1$ vertices. The benefits of such construction are twofolded.

First, Kingman's coalescent had already been exploited by Addario-Berry and the author to describe near-maximum degrees in RRTs, [2, 7]. With the new procedure, we are able to extract finer information about extreme degree values in RRTs. Second, growth procedures naturally couple families of trees as the size varies. However, typically there is no simple coupling of finite n -coalescent processes as n varies. The introduction of the Robin-Hood pruning provides, to the best of our knowledge, a novel tree growth procedure which is interesting on its own; thereby, opening a wide range of further avenues of research.

The Robin-Hood pruning is best described through an auxiliary tree structure that relates to both Kingman's coalescent and RRTs. We proceed to its description, then we present the results obtained in this work.

1.1. Notation. We denote natural logarithms by $\ln(\cdot)$ and logarithms with base 2 by $\log(\cdot)$. For $n \in \mathbb{N}$, we write $[n] = \{1, \dots, n\}$ and let \mathcal{S}_n be the set of permutations on $[n]$.

Given a rooted labeled tree $t = (V(t), E(t))$, write $|t| = |V(t)|$ and call $|t|$ the size of t . We write \mathcal{T}_n for the set of rooted trees t with vertex set $V(t) = [n]$. By convention, we direct all edges toward the root $r(t)$ and write $e = uv$ for an edge with tail u and head v . For $u \in V(t) \setminus \{r(t)\}$ we write $p_t(u)$ for the parent of u , that is, the unique vertex v with uv in $E(t)$. Finally, write $d_t(v)$ for the number of edges directed toward v in t , and call $d_t(v)$ the degree of v . Note that $d_t(v) = \#\{u : p_t(u) = v\}$.

We say $t \in \mathcal{T}_n$ is *increasing* if its vertex labels increase along root-to-leaf paths; in other words, if $t \in \mathcal{T}_n$ and $p_t(v) < v$ for all $v \in [n] \setminus \{r(t)\}$ (in particular, $r(t) = 1$). We write $\mathcal{I}_n \subset \mathcal{T}_n$ for the set of increasing trees of size n . It is easy to see that $|\mathcal{I}_n| = (n-1)!$ for all n . Next, a tree growth process is a sequence $(t_n, n \geq 1)$ of trees with $t_n \in \mathcal{T}_n$ for each n . The process is increasing if t_n is a subtree of t_{n+1} for all n ; this implies that $t_n \in \mathcal{I}_n$ for all n .

Tree growth processes select a tree from \mathcal{T}_n for each $n \geq 1$ usually with the characteristic that new vertices attach to some vertex in the previous tree, giving rise to increasing trees.

1.2. Recursively decorated trees. We begin with the standard construction of a RRT of size $n \geq 1$, which we denote by R_n . Start with R_1 as a single node with label 1. For each $1 < j \leq n$, R_j is obtained from R_{j-1} by adding a new vertex j and connecting it to $v_j \in [j-1]$; the choice of v_j is uniformly random and independent for each $1 < j \leq n$. The process $(R_n, n \geq 1)$ is a random increasing tree growth process. Moreover, it is readily seen that R_n is a random increasing tree uniformly chosen from \mathcal{I}_n .

Recursively decorated trees extend the concept of increasing trees. If $t \in \mathcal{T}_n$ and $\sigma \in \mathcal{S}_n$ then $\sigma(t)$ is the tree $t' \in \mathcal{T}_n$ with edges $\{\sigma(u)\sigma(v) : uv \in E(t)\}$. We say σ is an *addition history* for t if $\sigma(t)$ is increasing. If σ is an addition history for t then we say that the pair (t, σ) is a *recursively decorated tree* or *decorated tree*, and that vertex v has addition time $\sigma(v)$, for all $v \in V(t)$. Write

$$\mathcal{RD}_n = \{(t, \sigma) : t \in \mathcal{T}_n, \sigma \text{ is an addition history of } t\},$$

for the set of recursively decorated trees of size n . See Figure 1 for an example.

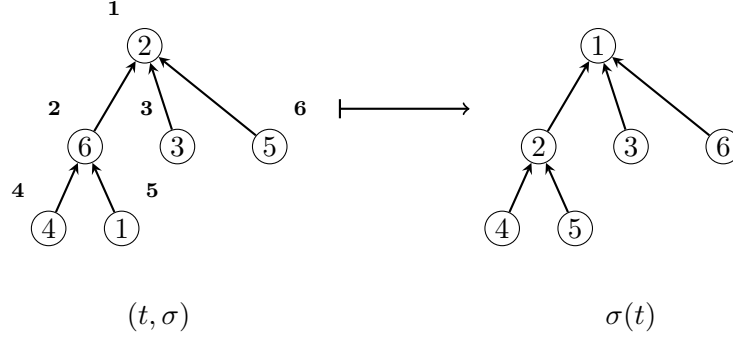


FIGURE 1. A decorated tree $(t, \sigma) \in \mathcal{RD}_6$ on the left; the permutation σ is depicted with bold numbers next to the vertices in t (so for example $\sigma(1) = 5$ and $\sigma(6) = 2$). On the right, the increasing tree $\sigma(t)$.

For each $n \geq 1$ let $\text{RT}_n = (T_n, \sigma_n)$ be a uniformly chosen decorated tree in \mathcal{RD}_n . We remark now that σ_n encodes the evolution of Kingman's coalescent on n vertices; the details on such correspondence are given in Section 5. The next, straightforward proposition shows that the shape of T_n has the same law as that of R_n .

Proposition 1.1. *For each $n \in \mathbb{N}$, $|\mathcal{RD}_n| = n!(n-1)!$ and if $\text{RT}_n = (T_n, \sigma_n) \in \mathcal{RD}_n$ is chosen uniformly at random then $\sigma_n(T_n)$ is a random recursive tree of size n and σ_n is a uniformly chosen permutation in \mathcal{S}_n .*

Proof. By definition, if $(t, \sigma) \in \mathcal{RD}_n$, then $\sigma(t) \in \mathcal{I}_n$. Let $\varphi : \mathcal{RD}_n \rightarrow \mathcal{I}_n \times \mathcal{S}_n$ be defined such that $\varphi(t, \sigma) = (\sigma(t), \sigma)$. For an increasing tree t and $\sigma \in \mathcal{S}_n$, let $t' = \sigma^{-1}(t)$ then $\varphi(t', \sigma) = (t, \sigma)$, it is also straightforward that φ is injective. Thus $|\mathcal{RD}_n| = |\mathcal{I}_n| \cdot |\mathcal{S}_n| = n!(n-1)!$. The result follows since bijections preserve the uniform measure on finite probability spaces. \square

Corollary 1.2. *For all $n \in \mathbb{N}$, the following distributional identity holds.*

$$(\text{d}_{\text{RT}_n}(\sigma_n^{-1}(v)); v \in [n]) \stackrel{\text{dist}}{=} (\text{d}_{R_n}(v); v \in [n]).$$

1.3. Statement of results. The Robin-Hood pruning $\text{RH}_n : \mathcal{RD}_{n-1} \rightarrow \mathcal{RD}_n$ is a random procedure that allows us to construct RT_n from RT_{n-1} while preserving most of the edges in RT_{n-1} .

Broadly speaking, $\text{RH}_n(t, \sigma)$ is obtained from (t, σ) by pruning some subtrees of t and placing them as subtrees of a new vertex labeled n ; additionally, vertex n attaches to a random vertex or becomes the root of the new tree. The addition history in $\text{RH}_n(t, \sigma)$ is adjusted from σ such that vertex n has a uniformly random *addition time*. Heuristically, the random procedure follows a ‘steal from the old to give to the new’ scheme; that is, once the addition time of n has been determined, vertices which had been added earlier (according to σ) have larger probability of being reattached to vertex n .

We will write $\text{RH} = \text{RH}_n$ when the size of the input is clear from the context. The exact definition of RH_n will be given in the next section along with the proof of the following result.

Theorem 1.3. *For each $n \geq 1$ let $\text{RT}_n = (T_n, \sigma_n)$ be a uniformly random element in \mathcal{RD}_n . The Robin-Hood pruning provides a coupling for $((T_n, \sigma_n), n \geq 1)$ by setting $(T_n, \sigma_n) = \text{RH}(T_{n-1}, \sigma_{n-1})$ for each $n \geq 2$.*

As we will establish in Proposition 5.3, RT_n is a representation of Kingman's coalescent on $[n]$. Therefore, we have the following corollary.

Corollary 1.4. *The construction of $((T_n, \sigma_n), n \geq 1)$ in Theorem 1.3 gives an explicit coupling of all finite Kingman's coalescents.*

A remarkable property of the coupling in Theorem 1.3 is that, it yields a tree growth process where all the trees are distributed as RRTs, however the process itself is not increasing. To the best of our knowledge, this is a novel evolution of random networks. Potential applications and open problems are discussed in Section 6.

Turning to extreme values in the degree sequence of RRTs, consider the following variables. For integers $0 < m \leq n$, let

$$X_m^{(n)} = \#\{v \in [n] : d_{R_n}(v) = m\};$$

Janson established the joint limiting distribution of $(X_m^{(n)}, m \geq 1)$ in [12]; for previous results on the degree distribution of RRTs see the survey [18]. In terms of capturing the degree sequence around the maximum degree range, Addario-Berry and the author provide all the possible limiting distributions of $(X_{\lfloor \log n \rfloor + k}^{(n)}, k \in \mathbb{Z})$ in [2].

In this work we are concerned with high-degree vertices in a broader sense; that is, we consider the variables

$$Z_m^{(n)} = \#\{v \in [n] : d_{R_n}(v) \geq m\},$$

with $m \sim c \ln n$ and provide results on convergence rates toward their limiting distributions. Throughout the paper, we write $\lambda_{n,m} = \mathbf{E} \left[Z_m^{(n)} \right]$ and record the following estimate.

Lemma 1.5 (Lemma 4.3 in [2]). *First, $\lambda_{n,m} \leq 2^{-m+\log n}$, and for each $c \in (0, 2)$, there is $\gamma(c)$ such that, uniformly over $m < c \ln n$,*

$$\lambda_{n,m} = n \mathbf{P}(d_{\text{RT}_n}(1) \geq m) = 2^{-m+\log n} (1 + o(n^{-\gamma})).$$

Our first result on high-degree vertices is obtained by applying the Chen-Stein method.

Theorem 1.6. *Fix $1 < c < c' < 2$. There are constants $\alpha = \alpha(c') \in (0, 1)$ and $\beta = \beta(c) > 0$ such that uniformly for $m = m(n)$ satisfying $c \ln n < m < c' \ln n$,*

$$d_{\text{TV}} \left(Z_m^{(n)}, \text{Poi}(\lambda_{n,m}) \right) \leq O(2^{-m+(1-\alpha)\log n}) + O(n^{-\beta}).$$

The exponent $-m + (1 - \alpha) \log n$ in Theorem 1.6 is negative when $(1 - \alpha) \log e < c$. A detailed but simple track of the conditions on α , see Proposition 3.2, shows that there is a non-empty interval $I_{c'} = ((1 - \alpha) \log e, c')$ such that if $c \in I_{c'} \cap (1, 2)$, then the bounds in Theorem 1.6 are, in fact, tending to zero. Moreover, by Lemma 1.5, if $c < \log e$, then $\lambda_{n,c \ln n} \rightarrow \infty$ as $n \rightarrow \infty$. This fact, together with Theorem 1.6 yields the next corollary.

Corollary 1.7. *For each $c' \in (1, \log e)$ there exists $c \in (1, c')$ such that if $c \ln n < m < c' \ln n$, then*

$$\frac{Z_m^{(n)} - \lambda_{n,m}}{\sqrt{\lambda_{n,m}}} \xrightarrow{\text{dist}} N(0, 1).$$

Corollary 1.7 extends the range of $m = m(n)$ for which a central limit theorem exists for $Z_m^{(n)}$; previous results were given for m constant [12] and for $m = \log n - d$ with $d = d(n)$ slowly tending to infinity [2].

We opted to state Theorem 1.6 and Corollary 1.7 separately to clarify where the bounds on m are limiting the convergence rates. In Section 3, we explain how previous results in [2] determine the exponent α , while the exponent β depends on an auxiliary coupling based on the Robin-Hood pruning. The details of such coupling are given in Section 4; for the moment we remark that, by Corollary 1.2, for all $m \leq n$,

$$(1) \quad Z_m^{(n)} \stackrel{\text{dist}}{=} \#\{v \in [n] : d_{\text{RT}_n}(v) \geq m\};$$

when there is no ambiguity, we write RT_n to refer only to its tree coordinate. The pruning procedure provides a key description of $(d_{\text{RT}_n}(i), i \in [n])$ in terms of both $(d_{\text{RT}_{n-1}}(i), i \in [n-1])$ and $d_{\text{RT}_n}(n)$, which is independent of the former vector. This allows us to analyze the conditional law of $(d_{\text{RT}_n}(i), i \in [n-1])$ given that $d_{\text{RT}_n}(n) \geq m$.

Our last result concerns the maximum degree Δ_n of R_n , which by Corollary 1.2 satisfies $\Delta_n \stackrel{\text{dist}}{=} \max\{d_{\text{RT}_n}(v) : v \in [n]\}$.

Theorem 1.8. *There exists $C > 0$ such that uniformly over $0 < i = i(n) < \log e \ln \ln n - C$,*

$$\mathbf{P}(\Delta_n < \lfloor \log n \rfloor - i) = \exp\{-2^{i+\varepsilon_n}\}(1 + o(1)),$$

where $\varepsilon_n = \log n - \lfloor \log n \rfloor$.

The first bounds of this type, for $\mathbf{P}(\Delta_n < \lfloor \log n \rfloor + j)$ with $j \in \mathbb{Z}$, were given in [9]. An extension to $j < 2 \ln n - \log n$ was obtained in [2]. Goh and Schmutz provide a heuristic of how the Gumbel, or double-exponential, distribution arises for Δ_n [9]. They do so by looking at the limiting distribution of $d_{R_n}(i)$ with $i \rightarrow \infty$ slowly. Below we present a distinct heuristic in terms of the degree distribution of RT_n .

The maximum of i.i.d. random variables is, under rather general conditions, distributed in the limit as the Gumbel distribution [11]. Lattice distributions are excluded from this regime and, instead, Anderson gives analogous conditions under which the Gumbel distribution serves as an approximation for their maximum [3]. In the case of $(\deg_{\text{RT}_n}(v), v \in [n])$, their limiting distributions are geometric, a distribution which satisfies the conditions given in [3]. Although, the degrees of a tree are not independent, their correlations are weak and the Gumbel-type approximation still arises for the distribution of Δ_n .

Outline. The remainder of the paper is organized as follows. The proof of Theorem 1.3 is given in the next section, and its connection to Kignman's coalescent in Section 5. In Section 3, we explain how we apply the Chen-Stein method to $Z_m^{(n)}$ by constructing an auxiliary coupling; the proofs of Theorems 1.6 and 1.8 and Corollary 1.7 are provided in Section 3 under the assumption of the existence of such coupling. The auxiliary coupling is based on the Robin-Hood pruning and is defined in Section 4. To close this work, we briefly discuss further avenues of research in Section 6.

2. THE ROBIN-HOOD PRUNING

For each $n \geq 2$, the Robin-Hood pruning RH_n is a random procedure that takes a decorated tree $(t, \sigma) \in \mathcal{RD}_{n-1}$ and outputs a decorated tree $\text{RH}_n(t, \sigma) \in \mathcal{RD}_n$. We first define a deterministic pruning, which is illustrated in Figure 2.

2.1. A deterministic process. For $d \geq 1$, we write $x = (x_1, \dots, x_d) \in \{0, 1\}^d$. Let $n > 1$ and set

$$\mathcal{C}_n = \{(k, l, x) : 1 \leq l < k \leq n, x \in \{0, 1\}^{n-1}\} \cup \{(1, 0, x) : x \in \{0, 1\}^{n-1}, x_1 = 1\};$$

additionally, for $(k, l, x) \in \mathcal{C}_n$ and a permutation $\sigma \in \mathcal{S}_{n-1}$, let

$$\mathcal{V}_n(k, l, x, \sigma) = \mathcal{V}_n(k, x, \sigma) = \{v \in [n-1] : x_{\sigma(v)} = 1, \sigma(v) \geq k\}.$$

Note that the definition of \mathcal{C}_n is such that $\sigma^{-1}(1) \in \mathcal{V}_n$ if and only if $k = 1$. The set \mathcal{V}_n corresponds to the vertices to be pruned and rewired in the following deterministic *pruning*.

Definition 2.1. For $n \geq 2$, $(t, \sigma) \in \mathcal{RD}_{n-1}$ and $(k, l, x) \in \mathcal{C}_n$ define $(t', \sigma') \in \mathcal{T}_n \times \mathcal{S}_n$ as follows.

First, t' is obtained from t as follows. Let $\mathcal{V} = \mathcal{V}_n(k, x, \sigma)$. For each $v \in \mathcal{V} \setminus \{r(t)\}$, replace the edge $vp_t(v)$ with an edge connecting v to a new vertex labeled n . Now, if $k = 1$ then attach $r(t)$ to n ; otherwise, attach vertex n to $\sigma^{-1}(l)$. In other words, the edges of t' are given by

$$E(t') = \begin{cases} (E(t) \cup \{vn; v \in \mathcal{V}\}) \setminus \{vp_t(v); v \in \mathcal{V}\} & \text{if } k = 1, \\ \{n\sigma^{-1}(l)\} \cup (E(t) \cup \{vn; v \in \mathcal{V}\}) \setminus \{vp_t(v); v \in \mathcal{V}\} & \text{if } k > 1. \end{cases}$$

Second, let $\sigma' : [n] \rightarrow [n]$ be defined by $\sigma'(n) = k$ and for $v < n$,

$$\sigma'(v) = \sigma(v) + \mathbf{1}_{[\sigma(v) \geq k]}.$$

We write $\text{rh}_n((t, \sigma), (k, l, x)) = (t', \sigma')$.

Lemma 2.2. Fix $n \geq 2$. For any $(t, \sigma) \in \mathcal{RD}_{n-1}$ and $(k, l, x) \in \mathcal{C}_n$,

$$\text{rh}_n((t, \sigma), (k, l, x)) \in \mathcal{RD}_n.$$

Proof. Write $\text{rh}_n((t, \sigma), (k, l, x)) = (t', \sigma')$. When $k = 1$, it is clear that t' is a tree. When $k > 1$, let $w = \sigma^{-1}(l)$ be the parent of n in t' and let $(w = v_1, \dots, v_j = r(t))$ be the path from w to the root of t . Since σ is an addition history of t ,

$$l = \sigma(v_1) > \sigma(v_2) > \dots > \sigma(v_j) = 1;$$

moreover, $l < k$. It follows that $v_i \notin \mathcal{V}(k, l, x, \sigma)$ for $i \in [j]$ and consequently, no edges in the path from n to the root in t' closes a cycle by connecting to n .

Now, we show that σ' is an addition history for t' . It is clear that σ' is a permutation of $[n]$, so it suffices to prove that $\sigma'(v) > \sigma'(p_{t'}(v))$, for all $v \in V(t) \setminus \{r(t')\}$. First, for vertices v with $p_{t'}(v) = n$ we have $\sigma(v) \geq k$ and consequently

$$\sigma'(v) = \sigma(v) + 1 > k = \sigma'(n).$$

Second, consider $v, w < n$ with $p_{t'}(v) = w$. It follows that $vw \in E(t)$ and thus $\sigma(v) > \sigma(w)$. Consequently, $\mathbf{1}_{[\sigma(v) \geq k]} \geq \mathbf{1}_{[\sigma(w) \geq k]}$ and so $\sigma'(v) > \sigma'(w)$. The last case occurs when $k > 1$ and $p_{t'}(n) = w = \sigma^{-1}(l)$. We then have

$$\sigma'(n) = k > l = \sigma(w) = \sigma'(w). \quad \square$$

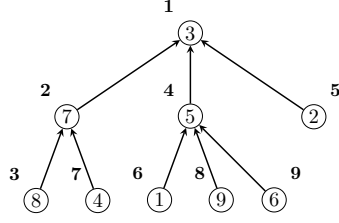
We note here a property of this pruning procedure that will be useful in the proof of Proposition 1.6; or more precisely, Proposition 4.3.

Fact 2.3. Fix $n \geq 2$. For any $(t, \sigma) \in \mathcal{RD}_{n-1}$ and $(k, l, x) \in \mathcal{C}_n$, write

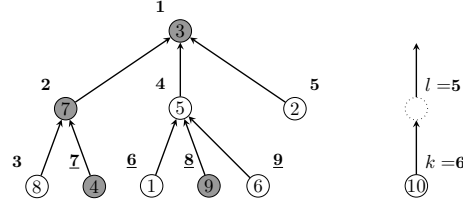
$$(t', \sigma') = \text{rh}_n((t, \sigma), (k, l, x)) \in \mathcal{RD}_n.$$

Then $d_{t'}(n) = \sum_{i=k}^{n-1} x_i$, and for $v \in [n-1]$,

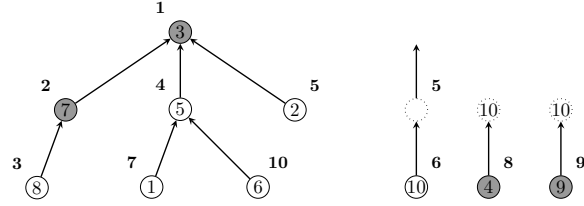
$$d_{t'}(v) = d_t(v) + \mathbf{1}_{[l=\sigma(v)]} - \sum_{i=k}^{n-1} x_i \mathbf{1}_{[v=p_t(\sigma^{-1}(i))]}.$$



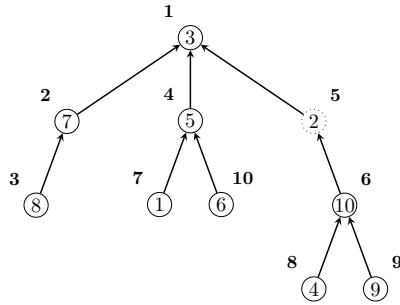
(A) A tree (t, σ) in \mathcal{RD}_9 . The permutation σ is depicted with bold numbers.



(B) In this case, $k = 6, l = 5$ and $x_1 = x_2 = x_7 = x_8 = 1$; all other $x_i = 0$. Vertices in gray satisfy $X_{\sigma(v)} = 1$ and underlined are addition times $\sigma(i) \geq k$.



(C) Nodes i with $\sigma(i) \geq k$ and $x_i = 1$ have been pruned and addition times have been adjusted.



(D) The resulting tree $\text{rh}_{10}((t, \sigma), (k, l, x)) \in \mathcal{RD}_{10}$.

FIGURE 2. An example of the Robin-Hood pruning for $n = 10$.

2.2. The random process. The Robin-Hood pruning is defined, for each $(t, \sigma) \in \mathcal{RD}_n$, by

$$\text{RH}_n(t, \sigma) = \text{rh}_n((t, \sigma), (K, L, X));$$

where the element $(K, L, X) \in \mathcal{C}_n$ is an RH_n set of random variables defined as follows.

Definition 2.4. Fix $n \geq 1$. Let $K \stackrel{\text{dist}}{=} \text{Unif}(1, 2, \dots, n)$; if $K = 1$ let $L = 0$, and if $K > 1$ let $L = \text{Unif}(1, 2, \dots, K-1)$. Independently, let $X_i = \text{Bernoulli}(1/i)$ be independent variables for $i \in [n-1]$ and write $X = (X_1, \dots, X_{n-1})$. An RH_n -set is a triple of random variables with the same law as (K, L, X) .

The law of $\text{RH}_n(t, \sigma)$ depends on the initial input (t, σ) ; however, the distribution of the RH_n -set of variables is defined so that $\text{RH}_n(\text{RT}_{n-1})$ preserves the uniform measure in decorated trees. To verify this claim, we require the following characterization of RT_n .

Lemma 2.5. Let $n \geq 1$ be an integer. A random decorated tree $(T, \sigma) \in \mathcal{RD}_n$ is uniformly random if and only if the following properties are satisfied.

- i) The permutation σ is uniformly random on \mathcal{S}_n .
- ii) The vertices

$$(p_T(v), v \in V(T) \setminus \{r(T)\}) = (p_{\sigma(T)}(\sigma^{-1}(v)), v \in V(T) \setminus \{r(T)\})$$

are, conditionally given σ , independent.

- iii) For all vertices $v, w \in [n]$ and indices $i, j \in [n]$,

$$(2) \quad \mathbf{P}(p_T(v) = w, \sigma(v) = j, \sigma(w) = i) = \frac{1}{n(n-1)(j-1)} \mathbf{1}_{[j>i]}.$$

Proof. Let (T, σ) be uniformly random on \mathcal{RD}_n . Then by Proposition 1.1, σ is a uniformly random permutation and $\sigma(T)$ has the law of R_n . Therefore, as multisets,

$$\{p_T(v), v \in V(T) \setminus \{r(T)\}\} \stackrel{\text{dist}}{=} \{p_{\sigma(T)}(v), 1 < v \leq n\};$$

and parents in RRTs are chosen independently for each of the vertices. The third condition follows immediately: For all $v, w, i, j \in [n]$, we obtain

$$\begin{aligned} \mathbf{P}(p_T(v) = w, \sigma(v) = j, \sigma(w) = i) &= \frac{1}{n(n-1)} \mathbf{P}(p_T(v) = w \mid \sigma(v) = j, \sigma(w) = i) \\ &= \frac{1}{n(n-1)} \mathbf{P}(p_{\sigma(T)}(j) = i) \\ &= \frac{1}{n(n-1)(j-1)} \mathbf{1}_{[j>i]}. \end{aligned}$$

Now consider a random decorated tree $(T, \sigma) \in \mathcal{RD}_n$ satisfying conditions i)-iii). Fix a decorated tree $(t, \pi) \in \mathcal{RD}_n$, and for $v \in V(t) \setminus \{r(t)\}$, let $w_v = p_t(v)$, then

$$(3) \quad \begin{aligned} \mathbf{P}(p_T(v) = w_v \mid \sigma = \pi) &= \mathbf{P}(p_T(v) = w_v \mid \sigma(v) = \pi(v), \sigma(w_v) = \pi(w_v)) \\ &= \frac{1}{\pi(w_v) - 1}. \end{aligned}$$

The first equality holds by condition ii) and the second by both i) and iii) since $\pi(v) > \pi(w_v)$.

Now, by definition, $\pi(t) \in \mathcal{I}_n$. The increasing tree t' is determined by the set of parents $\{p_{t'}(v), 1 < v \leq n\}$. Therefore,

$$\begin{aligned} \mathbf{P}(\sigma(T) = \pi(t) \mid \sigma = \pi) &= \mathbf{P}(p_{\sigma(T)}(v) = p_{\pi(t)}(v), 1 < v \leq n \mid \sigma = \pi) \\ &= \mathbf{P}(p_T(v) = p_t(v), v \in V(t) \setminus \{r(t)\} \mid \sigma = \pi) \\ &= \prod_{v \in V(t) \setminus r(T)} \mathbf{P}(p_T(v) = p_t(v) \mid \sigma = \pi) \\ &= [(n-1)!]^{-1}. \end{aligned}$$

Condition *ii*) gives the third equality; the last equality holds by (3) since

$$\{\pi(w_v), v \in V(t) \setminus \{r(t)\}\} = \{2, \dots, n\}.$$

Finally condition *i*) and the computations above show that, regardless of the choice of $(t, \pi) \in \mathcal{RD}_n$, we have

$$\begin{aligned} \mathbf{P}((T, \sigma) = (t, \pi)) &= \mathbf{P}(\sigma(T) = \pi(t) \mid \sigma = \pi) \mathbf{P}(\sigma = \pi) \\ &= \frac{1}{n!} \mathbf{P}(\sigma(T) = \pi(t) \mid \sigma = \pi) = [n!(n-1)!]^{-1}. \end{aligned} \quad \square$$

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. Let $(T, \sigma) \in \mathcal{RD}_{n-1}$ be a uniformly random decorated tree. Let (K, L, X) be an RH_n set and let $(T', \pi) = \text{rh}((T, \sigma), (K, L, X))$. It suffices to show that (T', π) satisfies the properties in Lemma 2.5.

First, condition *i*) follows from the construction of π and the distributions of both K and σ . Second, once conditioning on π , which is equivalent to conditioning on both σ and K , we get

$$\begin{aligned} \{p_{T'}(v), v \in V(T') \setminus \{r(T')\}\} &= \{p_{T'}(v), 1 < \pi(v) < \pi(n)\} \\ &\quad \cup \{p_{T'}(v), \pi(n) \leq \pi(v) \leq n\} \\ &= \{p_T(v), v \in 1 < \sigma(v) < K\} \\ &\quad \cup \{p_{T'}(v), (2 \vee K) \leq \pi(v) \leq n\}, \end{aligned}$$

where the last two sets are conditionally independent given π . Now, since (T, σ) is uniformly random in \mathcal{RD}_{n-1} , the parents $\{p_T(v), v \in 1 < \sigma(v) < K\}$ are independent, conditionally given σ (and thus, also conditionally given π). On the other hand, for v with $\pi(v) \geq K$,

$$p_{T'}(v) = \begin{cases} n & \text{if } X_{\pi(v)-1} = 1, \\ p_T(v) & \text{if } X_{\pi(v)} = 0, \\ \pi^{-1}(L) & \text{if } \pi(v) = K. \end{cases}$$

Note that $p_{T'}(v)$ is determined independently from other vertices, thus $\{p_{T'}(v), K \leq \pi(v) \leq n\}$ are also independent, conditionally given π . This implies that condition *ii*) is satisfied.

Third, fix $1 \leq i < j \leq n$ and fix distinct $v, w \in [n]$. We consider three cases; namely $v = n$, $w = n$, and $v, w \in [n-1]$. Let

$$\begin{aligned} A_1 &= \{p_{T'}(n) = w, \pi(n) = j, \pi(w) = i\}, \\ A_2 &= \{p_{T'}(v) = n, \pi(v) = j, \pi(n) = i\}, \\ A_3 &= \{p_{T'}(v) = w, \pi(v) = j, \pi(w) = i\}. \end{aligned}$$

It remains to show that the probabilities of A_1, A_2, A_3 are given by (2) for all $i, j \in [n]$. The event $p_{T'}(n) = w$ implies that $\sigma(w) = L < K$. Therefore, A_1 occurs precisely when $K = j$, $L = i$, and $\sigma(w) = i$. Then,

$$\mathbf{P}(A_1) = \mathbf{P}(K = j, L = i) \mathbf{P}(\sigma(w) = i) = \frac{1}{n(j-1)(n-1)}.$$

Next, $p_{T'}(v) = n$ implies that $\sigma(v) \geq K$ and thus $\pi(v) = \sigma(v) + 1$. It then follows that A_2 occurs when $K = i$, $\sigma(v) = j - 1$, and $X_{j-1} = 1$. Therefore,

$$\mathbf{P}(A_2) = \mathbf{P}(K = i, X_{j-1} = 1) \mathbf{P}(\sigma(v) = j - 1) = \frac{1}{n(j-1)(n-1)}.$$

For the last case, since $u, v < n$, it follows that $K \notin \{i, j\}$. For each $k \in [n] \setminus \{i, j\}$ let

$$A_{3,k} = \{p_{T'}(v) = w, \pi(v) = j, \pi(w) = i, K = k\}.$$

In computing the probabilities $\mathbf{P}(A_{3,k})$ we use that (T, σ) is uniformly random in RD_{n-1} . If $K > j$, then both $\sigma(v) = \pi(v)$ and $\sigma(w) = \pi(w)$; in addition, $p_{T'}(v) = w$ only if $p_T(v) = w$. Therefore, if $k > j$, then

$$\begin{aligned} \mathbf{P}(A_{3,k}) &= \mathbf{P}(K = k) \mathbf{P}(p_T(v) = w, \sigma(v) = j, \sigma(w) = i) \\ &= \frac{1}{n(n-1)(n-2)(j-1)}. \end{aligned}$$

Similarly, if $K < j$, then $\sigma(v) = \pi(v) - 1$, $\sigma(w) = \pi(w) - \mathbf{1}_{[K < i]}$, and additionally $X_{j-1} = 0$. It then follows that, if $k < j$,

$$\begin{aligned} \mathbf{P}(A_{3,k}) &= \mathbf{P}(K = k, X_{j-1} = 0) \mathbf{P}(p_T(v) = w, \sigma(v) = j - 1, \sigma(w) = i - \mathbf{1}_{[K < i]}) \\ &= \frac{1}{n} \cdot \frac{j-2}{j-1} \cdot \frac{1}{(n-1)(n-2)(j-2)}. \end{aligned}$$

We have shown that $\mathbf{P}(A_{3,k})$ is uniform for all $k \in [n] \setminus \{i, j\}$, and we get

$$\mathbf{P}(A_3) = \sum_{k \neq i, j} \mathbf{P}(A_{3,k}) = \frac{1}{n(n-1)(j-1)}.$$

Altogether, we have shown that condition *iii*) is satisfied and so the proof is complete. \square

3. LARGE DEGREES IN RRTs

The aim in this section is to bound the convergence rate of the law of ²

$$Z_m^{(n)} \stackrel{\text{dist}}{=} \#\{v \in [n] : d_{\text{RT}_n}(v) \geq m\}$$

to a suitable Poisson random variable. Our tool in this section is the Chen-Stein method as stated in Proposition 3.1 below. Given probability measures μ and ν , a coupling of μ and ν is a pair (X, Y) of random variables (either real or vector-valued) with $X \sim \mu$ and $Y \sim \nu$.

Let $I = (I_a, a \in \mathcal{A})$ be a collection of $\{0, 1\}$ -valued random variables. Let μ be the law of $W = \sum_{a \in \mathcal{A}} I_a$ and for $a \in \mathcal{A}$ let ν_a be the conditional law of W given that $I_a = 1$, so

$$\nu_a(B) = \mathbf{P}(W \in B \mid I_a = 1).$$

²By Fact 1.2, considering either R_n or RT_n in the definition of $Z_m^{(n)}$ is equivalent.

Proposition 3.1 ([10, Theorem 3.7]). *Let $I = (I_a, a \in \mathcal{A})$ be a collection of $\{0, 1\}$ -valued random variables. For each $a \in \mathcal{A}$ fix a coupling (W, W_a) of μ and ν_a . Then with $\lambda = \mathbf{E}[W]$, we have*

$$d_{\text{TV}}(W, \text{Poi}(\lambda)) \leq \min\{\lambda^{-1}, 1\} \sum_{a \in \mathcal{A}} \mathbf{E}[I_a] \mathbf{E}[|W - (W_a - 1)|].$$

If the variables $I = (I_a, a \in \mathcal{A})$ are exchangeable, then for any fixed $a \in \mathcal{A}$ and coupling (W, W_a) of μ and ν_a . Then

$$(4) \quad d_{\text{TV}}(W, \text{Poi}(\lambda)) \leq \mathbf{E}[|W - (W_a - 1)|].$$

Now, for the remainder of the section, fix m and let $I = (I_v, v \in [n])$ have $I_v = \mathbf{1}_{[d_{\text{RT}_n}(v) \geq m]}$; in that case, $W = \sum_{i \in [n]} I_v = Z_m^{(n)}$. The next proposition, which states that the random variables (I_1, \dots, I_n) are ‘nearly’ negatively correlated, is an important input to the proof of Theorem 1.6.

Proposition 3.2. *For any $c \in (0, 2)$ there exists $\alpha = \alpha(c) > 0$ such that uniformly for $m = m(n) < c \ln n$ and distinct $v, w \in [n]$,*

$$\mathbf{E}[I_v I_w] - \mathbf{E}[I_v] \mathbf{E}[I_w] \leq O(2^{-2m - \alpha \log n}).$$

Moreover, $\alpha < \frac{1}{4}(1 - c + \sqrt{1 + 2c - c^2}) < 1$.

The proof of Proposition 3.2 appears in Appendix A; we make precise the upper bound for α in Proposition 3.2 as this is crucial to Corollary 1.7. We note that a weaker version of Proposition 3.2, without explicit error bounds, was proved in [2, Proposition 4.2].

Additionally, we note in passing that the degree sequence of RRTs $(d_{R_n}(v), v \in [n])$ is negative orthant dependent; for a definition see [5]. This fact can be proven by induction from the two-vertex case $(d_{R_n}(v), d_{R_n}(w))$, which, in turn, follows essentially from the negative orthant dependency of multinomial distributions, see e.g. [4, Lemma 1]. As a consequence, for all $v, w \in [n]$,

$$(5) \quad \mathbf{P}(d_{R_n}(v) \geq m, d_{R_n}(w) \geq m) - \mathbf{P}(d_{R_n}(v) \geq m) \mathbf{P}(d_{R_n}(w) \geq m) \leq 0.$$

In a slight abuse of notation let us denote by μ the law of (I_1, \dots, I_n) and denote by $\nu = \nu_n$ the conditional law of (I_1, \dots, I_n) given that $I_n = 1$. Now, let $(I, J) = ((I_v, v \in [n]), (J_v, v \in [n]))$ be a coupling of μ and $\nu = \nu_n$ and write $W_n = \sum_{v \in [n]} J_v$, we get

$$(6) \quad \mathbf{E}[|W - (W_n - 1)|] \leq \mathbf{E}[I_n] + \sum_{v \in [n-1]} \mathbf{E}[|I_v - J_v|].$$

To apply Proposition 3.1 with as tight as possible bounds, one has to analyze couplings of μ and ν . For example, suppose one can provide an explicit coupling such that $J_v - I_v \leq 0$ almost surely, for all $v \in [n - 1]$. It would then follow that

$$\mathbf{E}[I_n I_v] - \mathbf{E}[I_n] \mathbf{E}[I_v] = -\mathbf{E}[I_n] \mathbf{E}[|J_v - I_v|] \leq 0;$$

which would be the corresponding inequality to (5). Although we do not claim the bounds in Proposition 3.2 are optimal, it seems that the property in (5) is lost when randomizing the vertex labels of R_n to obtain RT_n .

Nevertheless, Proposition 3.2 suggests we can provide couplings of μ and ν for which $I_v - J_v \geq 0$ for all $v \in [n - 1]$ with high probability. The next proposition is the key ingredient in using Proposition 3.1 to prove Theorem 1.6.

Proposition 3.3. *Let $c \in (1, 2)$. There is $\beta = \beta(c) > 0$ such that for any $m = m(n) > c \ln n$ there exists a coupling $(I, J) = ((I_1, \dots, I_n), (J_1, \dots, J_n))$ of μ and ν , in which for all $v \in [n-1]$,*

$$\mathbf{P}(I_v < J_v) \leq O(n^{-1-\beta}).$$

The coupling of Proposition 3.3 is based on the Robin-Hood pruning and its proof is the content of Section 4. Next, we provide the proofs of Theorem 1.6 (assuming Proposition 3.3), followed by the proofs of Corollary 1.7 and Theorem 1.8.

Proof of Theorem 1.6 assuming Proposition 3.3. Fix $1 < c < c' < 2$ and let $c \ln n < m = m(n) < c' \ln n$. We apply the Chen-Stein method to $Z_m^{(n)} \stackrel{\text{dist}}{=} \sum_{v \in [n]} I_v$. First, we use the coupling $(I, J) = ((I_1, \dots, I_n), (J_1, \dots, J_n))$ of μ and ν given in Proposition 3.3. By (4) and (6), we have

$$\text{d}_{\text{TV}} \left(Z_m^{(n)}, \text{Poi}(\mathbf{E}[\lambda_{n,m}]) \right) \leq \mathbf{E}[I_n] + \sum_{v \in [n-1]} \mathbf{E}[|I_v - J_v|].$$

It thus remains to show that the terms in the bound above are $O(2^{-m+(1-\alpha)\log n}) + O(n^{-\beta})$, where $\alpha = \alpha(c') \in (0, 1)$ and $\beta = \beta(c) > 0$ are defined as in Propositions 3.2 and 3.3. First, by Lemma 1.5 and the fact that $\alpha < 1$ gives

$$\mathbf{E}[I_n] = 2^{-m}(1 + o(1)) = O(2^{-m+(1-\alpha)\log n}).$$

Second, for any $v \in [n-1]$,

$$\begin{aligned} \mathbf{E}[I_n] \mathbf{E}[|J_v - I_v|] &= \mathbf{E}[I_n] \mathbf{E}[I_v - J_v] + 2\mathbf{E}[I_n] \mathbf{E}[(J_v - I_v)\mathbf{1}_{[I_v < J_v]}] \\ &= (\mathbf{E}[I_n] \mathbf{E}[I_v] - \mathbf{E}[I_n I_v]) + 2\mathbf{E}[I_n] \mathbf{P}(I_v < J_v). \end{aligned}$$

The terms in the last line are bounded by Proposition 3.2 and Proposition 3.3, respectively; giving

$$\begin{aligned} \sum_{v \neq n} \mathbf{E}[|I_v - J_v|] &= \frac{n-1}{\mathbf{E}[I_n]} ((\mathbf{E}[I_n] \mathbf{E}[I_v] - \mathbf{E}[I_n I_v]) + 2\mathbf{E}[I_n] \mathbf{P}(I_v < J_v)) \\ &\leq \frac{n}{\mathbf{E}[I_n]} (O(2^{-2m-\alpha\log n}) + \mathbf{E}[I_n] O(n^{-1-\beta})) \\ &= O(2^{-m+(1-\alpha)\log n}) + O(n^{-\beta}). \end{aligned}$$

In the last line we also use that Lemma 1.5 implies that $\mathbf{E}[I_n]^{-1} = O(2^m)$. \square

Proof of Corollary 1.7. Fix $c' \in (1, \log e)$ and let $\alpha = \alpha(c')$ be as in Theorem 1.6. Simple computations using the upper bound in Proposition 3.2 for α show that $(1-\alpha)\log e < c'$. Thus, we can chose $c \in ((1-\alpha)\log e, c')$.

Let $m = m(n)$ be such that $c \ln n < m < c' \ln n$. By the choice of c and c' , we have that, as $n \rightarrow \infty$, $(1-\alpha)\log n - m < 0$; while by Lemma 1.5,

$$\mathbf{E}[Z_m^{(n)}] = 2^{-m+\log n}(1 + o(1)) \rightarrow \infty.$$

The result then follows by Theorem 1.6 and the central limit theorem of Poisson variables, see e.g. [6, Exercise 3.4.4]. \square

Proof of Theorem 1.8. Recall that $\varepsilon_n = \log n - \lfloor \log n \rfloor$. Let $i = i(n)$ satisfy $0 < i < \log e \ln \ln n - C$, where $C > 0$ is a constant to be determined below, and note that $2^{i+\varepsilon_n} \leq 2^{i+1} < 2^{-C+1} \ln n$. Let $m = \lfloor \log n \rfloor - i$ and $Z \stackrel{\text{dist}}{=} \text{Poi}(\lambda_{m,n})$.

We have that $\{\Delta_n < \lfloor \log n \rfloor - i\}$ if and only if $\{Z_m^{(n)} = 0\}$. Therefore,

$$(7) \quad \mathbf{P}(\Delta_n < \lfloor \log n \rfloor - i) = \mathbf{P}(Z_m^{(n)} = 0) \leq \mathbf{P}(Z = 0) + d_{\text{TV}}(Z_m^{(n)}, Z).$$

We deal with the two terms on the right-hand side of (7) separately. First, using the lower bound on i , there is a constant $c \in (\log e, 2)$ such that for n large enough, $m - i < c \ln n$. Therefore, Lemma 1.5 gives $\gamma > 0$ such that $\lambda_{n,m} = 2^{i+\varepsilon_n} + o(n^{-\gamma} \ln n)$. Consequently,

$$\mathbf{P}(Z = 0) = \exp\{-\lambda_{n,m}\} = \exp\{-2^{i+\varepsilon_n}\}(1 + o(1)).$$

For the second term in (7), Theorem 1.6 gives $\alpha, \beta > 0$ such that

$$d_{\text{TV}}(Z_m^{(n)}, Z) = O(2^{-m+(1-\alpha)\log n}) + O(n^{-\beta}).$$

It remains to deal with these error terms. Note that $\exp\{2^{i+\varepsilon_n}\} \leq \exp\{2^{-C+1} \ln n\}$. Therefore, if $C > 1 + \log(1/\beta)$ then

$$\exp\{2^{i+\varepsilon_n}\} O(n^{-\beta}) = O(\exp\{(2^{-C+1} - \beta) \ln n\}) \rightarrow 0;$$

similarly, for C large enough,

$$\exp\{2^{i+\varepsilon_n}\} O(2^{-m+(1-\alpha)\log n}) = \exp\{2^{i+\varepsilon_n}\} O(2^{i-\alpha\log n}) \rightarrow 0.$$

The two limits above imply that $d_{\text{TV}}(Z_m^{(n)}, Z) = o(\exp\{-2^{i+\varepsilon_n}\})$, completing the proof. \square

4. PROOF OF PROPOSITION 3.3: AN AUXILIARY COUPLING

In this section we fix $n \in \mathbb{N}$, $c \in (1, 2)$ and $m = m(n) > c \ln n$. Let (T, σ) be uniformly random in \mathcal{RD}_{n-1} . Consider (K, L, X) an RH_n -set and let (K', L', X') be distributed as an RH_n -set conditioned to satisfy $\sum_{i=K}^{n-1} X'_i \geq m$. Now, write

$$(8) \quad (T', \pi) = \text{rh}((T, \sigma), (K, L, X)) = \text{RH}(T, \sigma),$$

$$(9) \quad (T'_m, \pi) = \text{rh}((T, \sigma), (K', L', X')).$$

By Fact 2.3, (T'_m, π) is a conditional version of $\text{RH}_n(T, \sigma)$ given that $d_{\text{RH}_n(T, \sigma)}(n) \geq m$. Consequently, any coupling of (K, L, X) and (K', L', X') yields a coupling of the measures μ and ν , defined in Section 3, by setting $I_v = \mathbf{1}_{[d_{T'}(v) \geq m]}$ and $J_v = \mathbf{1}_{[d_{T'_m}(v) \geq m]}$ for all $v \in [n]$. With this notation, our goal is to couple (K, L, X) and (K', L', X') in such a way that for some $\beta = \beta(c) > 0$,

$$(10) \quad \mathbf{P}(d_{T'}(v) < m \leq d_{T'_m}(v)) = O(n^{-1-\beta}).$$

We start with some straightforward lemmas. For any integer $n - m \leq k < n$, let $X^k = (X_i^k, i \in [n-1])$ be a conditional version of X given that $\sum_{i=k}^{n-1} X_i \geq m$. The following observation is quite standard but we include a proof for completeness. For $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d) \in \{0, 1\}^d$, $a \leq b$ only if $a_i \leq b_i$ for all $i \in [d]$. We say that $S \subset \{0, 1\}^d$ is monotone if $a \leq b$ and $a \in S$ implies $b \in S$.

Lemma 4.1. *For each $k < n$, there exists a coupling of X^k and X such that $X_i \leq X_i^k$ for all $i \in [n-1]$.*

Proof. By Strassen's theorem [14], it suffices to prove that X^k dominates stochastically X . That is, for all monotone subsets $S \in \{0, 1\}^{n-1}$,

$$(11) \quad \mathbf{P}(X^k \in S) \geq \mathbf{P}(X \in S).$$

Note that $S_k = \{a \in \{0, 1\}^{n-1} : a_1 + \dots + a_k \geq m\}$ is a monotone subset of $\{0, 1\}^{n-1}$. By Harris inequality, for any monotone subset $S \in \{0, 1\}^{n-1}$,

$$\mathbf{P}(X \in S \cap S_k) \geq \mathbf{P}(X \in S_k) \mathbf{P}(X \in S).$$

Dividing the above inequality by $\mathbf{P}(X \in S_k)$ yields (11); thus, completing the proof. \square

Lemma 4.2. *There exists a coupling of (K, L) and (K', L') such that $K' \leq K$ and $L' \leq L$.*

Proof. Let U_1, U_2 have uniform distributions on $[0, 1]$. Set $K = \lceil nU_1 \rceil$, and $L = \lceil (K - 1)U_2 \rceil$. Independently of U_1 and U_2 , let $X = (X_1, \dots, X_{n-1})$ be independent with $X_i \stackrel{\text{dist}}{=} \text{Bernoulli}(1/i)$. Then (K, L, X) is an RH_n -set.

Next, for each $k \in [n]$, let $p_k = \mathbf{P}\left(K = k \mid \sum_{i=k}^{n-1} X_i \geq m\right)$ and set

$$K' = \max \left\{ k : U_1 > \sum_{j=1}^{k-1} p_j \right\}$$

and $L' = \lceil (K' - 1)U_2 \rceil$.

The random variable K' has the correct law by construction. Moreover, conditionally given $K' = k$,

$$L' \stackrel{\text{dist}}{=} \begin{cases} \text{Unif}(k-1) & \text{if } k > 1, \\ 0 & \text{if } k = 1; \end{cases}$$

note that the distribution of L conditionally given $K = k$ has the same expression. Therefore, for all $l \leq k-1$ we have

$$\begin{aligned} \mathbf{P}(K' = k, L' = l) &= \mathbf{P}(L' = l \mid K' = k) \mathbf{P}(K' = k) \\ &= \frac{\mathbf{P}(L = l, K = k) \mathbf{P}\left(K = k, \sum_{i=k}^{n-1} X_i \geq m\right)}{\mathbf{P}(K = k) \mathbf{P}\left(\sum_{i=k}^{n-1} X_i \geq m\right)} \\ &= \frac{\mathbf{P}\left(K = k, L = l, \sum_{i=K}^{n-1} X_i \geq m\right)}{\mathbf{P}\left(\sum_{i=K}^{n-1} X_i \geq m\right)} \\ &= \mathbf{P}\left(K = k, L = l \mid \sum_{i=K}^{n-1} X_i \geq m\right); \end{aligned}$$

the third equality holds by the independence between X and (K, L) . It follows that (K', L') has the correct law.

Finally, since X is independent of K , for each $k \in [n]$,

$$\mathbf{P}(K' = k) = \frac{\mathbf{P}\left(K = k, \sum_{i=k}^{n-1} X_i \geq m\right)}{\mathbf{P}\left(\sum_{i=K}^{n-1} X_i \geq m\right)} = \left[n \mathbf{P}\left(\sum_{i=K}^{n-1} X_i \geq m\right) \right]^{-1} \mathbf{P}\left(\sum_{i=k}^{n-1} X_i \geq m\right).$$

This chain of equalities show that p_k is proportional to $\mathbf{P}\left(\sum_{i=k}^{n-1} X_i \geq m\right)$, which is decreasing in k . Consequently, if $K' = j$ then

$$U_1 > \sum_{i=1}^{j-1} p_i \geq \frac{j-1}{n},$$

in other words, $K \geq j = K'$. In turn, $L = \lceil (K-1)U_2 \rceil \geq \lceil (K'-1)U_2 \rceil = L'$. \square

Proposition 4.3. *There exists a coupling of (K, L, X) and (K', L', X') such that $K \geq K'$, $L \geq L'$ and $X_i \leq X'_i$ for all $i \in [n-1]$.*

Proof. First, couple $((K, L), (K', L'))$ as in Lemma 4.2 and also let $X = (X_1, \dots, X_{n-1})$ be as in the proof of that lemma. For each $1 \leq k < n$ fix a vector X^k coupled with X according to Lemma 4.1.

The dependence structure of X^1, \dots, X^{n-1} is unimportant to the argument, but for concreteness we may, e.g., take them to be conditionally independent given X . On the other hand, it is important to insist that the X^i are independent of (K', L') . Since (K', L') are determined by uniform variables U_1, U_2 independent of X , the existence of such joint coupling is straightforward.

Next, for each $i \in [n-1]$ write $X'_i = X_i^{K'}$ and let $X' = (X'_1, \dots, X'_{n-1})$. Now it remains to show that, (K', L', X') has the correct law.

For any $(k, l, x) \in \mathcal{C}_n$ and $j \leq n-1$, we use the independence of X^j from (K', L') to obtain,

$$\begin{aligned} \mathbf{P}(X' = x, K' = k, L' = l) &= \mathbf{P}(X^k = x) \mathbf{P}(K' = k, L' = l) \\ &= \frac{\mathbf{P}\left(X = x, \sum_{i=k}^{n-1} x_i \geq m\right)}{\mathbf{P}\left(\sum_{i=k}^{n-1} x_i \geq m\right)} \cdot \frac{\mathbf{P}\left(K = k, L = l, \sum_{i=k}^{n-1} x_i \geq m\right)}{\mathbf{P}\left(\sum_{i=k}^{n-1} x_i \geq m\right)} \\ &= \frac{\mathbf{P}\left(X = x, K = k, L = l, \sum_{i=k}^{n-1} x_i \geq m\right)}{\mathbf{P}\left(\sum_{i=k}^{n-1} x_i \geq m\right)} \\ &= \mathbf{P}\left(X = x, K = k, L = l \left| \sum_{i=k}^{n-1} x_i \geq m\right.\right). \end{aligned}$$

That the coupling $((K, L, X), (K', L', X'))$ has the desired properties follows from Lemmas 4.1 and 4.2. \square

Under the coupling of Proposition 4.3 we obtain necessary conditions for $d_{T'}(v) < m \leq d_{T'_m}(v)$ to hold.

Lemma 4.4. *Using variables as in Proposition 4.3 and the trees defined in (8) and (9). For any $v \in [n-1]$,*

$$\{d_{T'}(v) < m \leq d_{T'_m}(v)\} \subset \{L' = \sigma(v)\} \cap \{d_T(v) \geq m-1\}.$$

Proof. From the properties of the coupling in Proposition 4.3,

$$(12) \quad \sum_{i=K}^{n-1} X_i \mathbf{1}_{[v=p_T(\sigma^{-1}(i))]} \leq \sum_{i=K'}^{n-1} X'_i \mathbf{1}_{[v=p_T(\sigma^{-1}(i))]}.$$

Consequently, using Fact 2.3 we have that $d_{T'_m}(v) - d_{T'}(v) \leq \mathbf{1}_{[L'=\sigma(v)]}$. On the other hand, if $\{d_{T'}(v) < m \leq d_{T'_m}(v)\}$ holds, then it follows that $d_{T'_m}(v) - d_{T'}(v) > 0$ and so it is necessary that $\{L' = \sigma(v)\}$ holds.

Finally, $\{d_{T'}(v) < m \leq d_{T'_m}(v)\}$ implies that

$$m \leq d_{T'_m}(v) = d_T(v) + \mathbf{1}_{[L'=\sigma(v)]} - \sum_{i=K'}^{n-1} X'_i \mathbf{1}_{[v=p_T(\sigma^{-1}(i))]} \leq d_T(v) + 1;$$

or equivalently, that $\{d_T(v) \geq m - 1\}$. \square

The next tail bounds the degree of vertices in RRTs are obtained using standard estimates for binomial variables.

Lemma 4.5. *Fix $c > 1$. There exists $\beta = \beta(c) > 0$ such that uniformly over $m > c \ln n$, and for each $i \in [n]$,*

$$\mathbf{P}(d_{R_n}(i) > m) = O(n^{-\beta}).$$

Proof. Let $(B_k, k \geq 1)$ be independent Bernoulli variables with mean $1/k$ respectively. By the construction of R_n we have that $d_{R_n}(i) \stackrel{\text{dist}}{=} \sum_{k=i}^n B_k \leq \sum_{k=1}^n B_k$. Therefore,

$$\mathbf{P}(d_{R_n}(i) > m) \leq \mathbf{P}(d_{R_n}(1) > m) \leq \mathbf{P}\left(\sum_{k=1}^n B_k > c \ln n\right).$$

We use the following version of Bernstein inequalities (see, e.g. [13] Theorem 2.8, (2.5)). For a sum S of $\{0, 1\}$ -valued variables and $\varepsilon > 0$,

$$\mathbf{P}(S - \mathbf{E}[S] > \varepsilon \mathbf{E}[S]) \leq \exp\left\{-\frac{3\varepsilon^2}{2(3+\varepsilon)} \mathbf{E}[S]\right\}.$$

Since $\mathbf{E}[\sum_{k=1}^n B_k] = \ln n + O(1) < c \ln n$, we can use the above inequality with $\varepsilon = c - 1 + o(1)$ and set $\beta = \frac{3\varepsilon^2}{2(3+\varepsilon)}$. \square

Proof of Proposition 3.3. Fix $c \in (1, 2)$. Let $m = m(n) > c \ln n$ and $\beta = \beta(c) > 0$ be as in Lemma 4.5. Let us use (T', T'_m) as defined in (8) and (9) with $((K, L, X), (K', L', X'))$ coupled as in Proposition 4.3. Set $I_v = \mathbf{1}_{[d_{T'}(v) \geq m]}$ and $J_v = \mathbf{1}_{[d_{T'_m}(v) \geq m]}$ for all $v \in [n]$. We now show that $(I, J) = ((I_1, \dots, I_n), (J_1, \dots, J_n))$ is a coupling of the measures μ and ν which satisfies

$$\mathbf{P}(I_n < J_n) = \mathbf{P}(d_{T'}(v) < m \leq d_{T'_m}(v)) = O(n^{-1-\beta}).$$

First, using Lemma 4.4, we have

$$\mathbf{P}(d_{T'}(v) < m \leq d_{T'_m}(v)) \leq \sum_{j=1}^{n-1} \mathbf{P}(\sigma(v) = j, L' = j, d_T(v) \geq m - 1).$$

On the other hand, σ is uniformly random in \mathcal{S}_{n-1} and L' is independent of σ . Therefore, uniformly for each $j \in [n - 1]$,

$$\begin{aligned} \mathbf{P}(\sigma(v) = j, L' = j, d_T(v) \geq m - 1) &= \frac{1}{n-1} \mathbf{P}(d_T(v) \geq m - 1 \mid \sigma(v) = j) \mathbf{P}(L' = j) \\ &= \frac{1}{n-1} \mathbf{P}(d_{R_{n-1}}(j) \geq m - 1) \mathbf{P}(L' = j) \\ &\leq \mathbf{P}(L' = j) O(n^{-1-\beta}); \end{aligned}$$

the second inequality, since $\sigma(T) \stackrel{\text{dist}}{=} R_{n-1}$ and the last one by Lemma 4.5. Therefore, we get for all $v \in [n-1]$,

$$\begin{aligned} \mathbf{P}(I_v < J_v) &\leq \sum_{j=1}^{n-1} \mathbf{P}(\sigma(v) = j, L' = j, d_T(v) \geq m-1) \\ &= O(n^{-1-\beta}) \sum_{j=1}^{n-1} \mathbf{P}(L' = j) = O(n^{-1-\beta}). \end{aligned}$$

□

5. COALESCENTS AS RECURSIVELY DECORATED TREES

Coalescent processes are essentially defined as Na and Rapoport described ‘static’ trees. We will first explain the definition of coalescents using chains of forests and decorated trees. Following, we define Kingman’s coalescent in terms of such chains and briefly note its connection with increasing binary trees.

A forest f is a set of trees with pairwise disjoint vertex sets. Denote by $V(f)$ and $E(f)$, respectively, the union of the vertex and edge sets of the trees in f . For $n \geq 1$, an n -chain is a sequence $C = (f_n, \dots, f_1)$ of elements of $\mathcal{F}_n = \{f : V(f) = [n]\}$ such that, for $1 < i \leq n$, f_{i-1} is obtained from f_i by adding a directed edge between the roots of some pair of trees in f_i . In particular, f_n consists of n one-vertex trees and f_1 consists of a single tree on n vertices denoted by t_C . For an example see Figure 3.

The relation of n -chains with coalescents is the following. For an n -chain (f_n, \dots, f_1) , each of the trees of f_i correspond to a set of coalesced elements after $n - i + 1$ steps of the process. At each step, two sets (represented by their roots) coalesce and a new representative is chosen.

To link n -chains with decorated trees, we first define a natural edge labeling that tracks the number of *trees left* in the forest when a give edge comes along. Fix $C = (f_n, \dots, f_1)$, for each $e \in E(t_C)$, let

$$\rho_C(e) = \max\{i \in [n-1] : e \in E(f_i)\}.$$

We next define a vertex labeling $\sigma_C : V(t_C) \rightarrow [n]$. For each $uv \in E(t_C)$, let

$$\sigma_C(u) = \rho_C(uv) + 1;$$

and let $\sigma_C(r(t_C)) = 1$. The pair $(t_C, \sigma_C) \in \mathcal{RD}_n$ contains all the information to recover the original n -chain C .

Proposition 5.1. *Let \mathcal{CF}_n be the set of n -chains and $\Upsilon : \mathcal{CF}_n \rightarrow \mathcal{RD}_n$ be defined as follows. For an n -chain $C = (f_n, \dots, f_1)$, let $\Upsilon(C) = (t_C, \sigma_C)$. Then Υ is a bijection.*

Proof. First, we show that \mathcal{CF}_n and \mathcal{RD}_n have the same cardinality and that Υ is well defined.

To count the number of n -chains, consider constructing (f_n, \dots, f_1) by deciding which edge to add from f_k to f_{k-1} . Since there are k trees in f_k , when we have chosen (f_1, \dots, f_k) , there are $k(k-1)$ possible directed edges to add. Therefore, $|\mathcal{CF}_n| = n!(n-1)!$.

Next, let $C = (f_n, \dots, f_1)$ be an n -chain. For each $1 \leq i < n$, the new edge in f_i joins the roots of two trees in f_{i+1} and is directed towards the root of the resulting tree. Thus, the labels $\{\rho_C(e), e \in E(t_C)\}$ decrease along all paths in t_C towards the root $r(t_C)$. Consequently, the labels $\{\sigma_C(v), v \in [n]\}$ are, indeed, an addition history of t_C .

Now we show that Υ is injective and, thus it is a bijection between \mathcal{CF}_n and \mathcal{RD}_n . Consider two distinct n -chains $C = (f_n, \dots, f_1)$ and $C' = (f'_n, \dots, f'_1)$, then $k = \min\{i :$

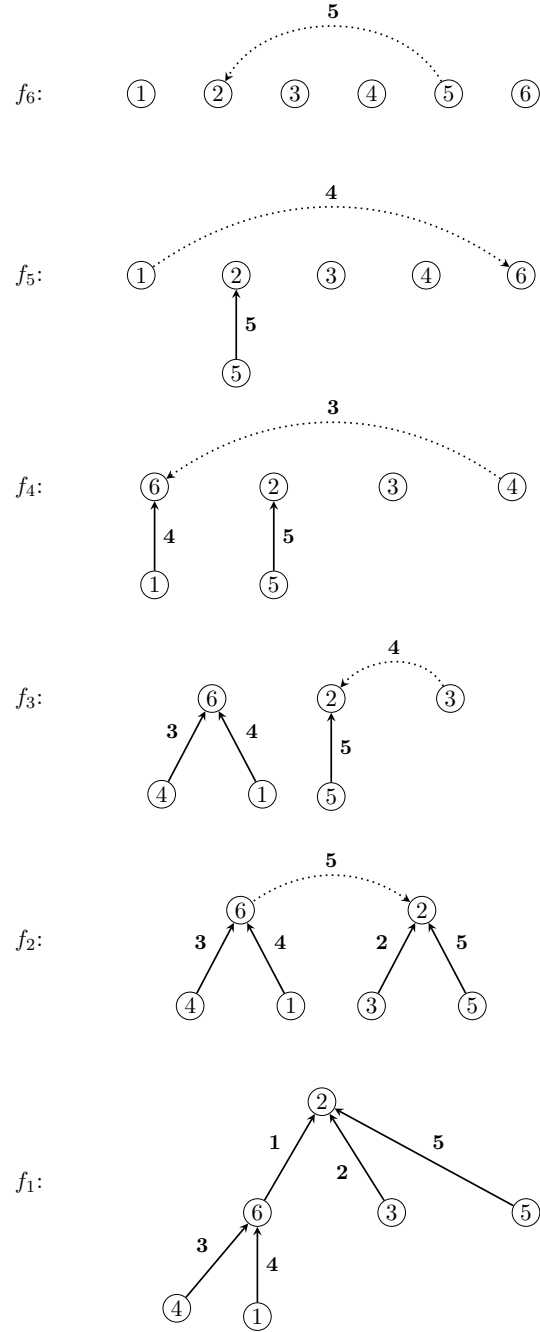


FIGURE 3. An example of an n -chain with $n = 6$. The edge labelling ρ_n is presented with numbers in bold.

$f_i \neq f'_i$ is well defined. If $k = 1$ then $t_C \neq t_{C'}$ and clearly, $\Upsilon(C) \neq \Upsilon(C')$. Otherwise, the edges $e \in E(f_{k-1}) \setminus E(f_k)$ and $e' \in E(f'_{k-1}) \setminus E(f'_k)$ are distinct. However $t_C = t_{C'}$, it thus follows that $e = uv \in f'_k$ and so $\sigma_C(u) = k > \sigma_{C'}(u)$. This shows that Υ is injective, completing the proof. \square

Kingman's coalescent is characterized by the property that the merging probability of any pair of components is independent of the components' sizes. The following definition describes Kingman's n -coalescent as a random n -chain $\mathbf{C} = (F_n, \dots, F_1)$. This construction has been previously exploited to study high-degree vertices in RRTs [2] and is closely related to the 'union-find' algorithm used in computer science (see e.g. [19]).

For an n -chain (f_n, \dots, f_1) and $1 \leq i \leq n$, list the trees of f_i in increasing order of their smallest-labeled vertex as $t_1^{(i)}, \dots, t_i^{(i)}$. Independently for each $1 < i \leq n$ let $\{a_i, b_i\} \subset \{\{a, b\} : 1 \leq a < b \leq i\}$ be uniformly chosen at random; in addition, let ξ_i be independent Bernoulli random variables with mean $1/2$.

Definition 5.2. *Kingman's n -coalescent is defined as $\mathbf{C} = (F_n, \dots, F_1)$ constructed as follows. For $1 < i \leq n$, F_{i-1} is obtained from F_i by adding an edge between $r(T_{a_i}^{(i)})$ and $r(T_{b_i}^{(i)})$. If $\xi_i = 1$ then direct the edge towards $r(T_{a_i}^{(i)})$; otherwise direct it towards $r(T_{b_i}^{(i)})$. The forest F_{i-1} consists of the new tree and the remaining $i - 2$ unaltered trees from F_i .*

Proposition 5.3. *Let $\mathbf{C} = (F_n, \dots, F_1)$ be a Kingman's coalescent and let $\text{RT}_n \in \mathcal{RD}_n$ be uniformly random. Then $T_{\mathbf{C}} \stackrel{\text{dist}}{=} \text{RT}_n$.*

Proof. Let $\mathbf{C} = (F_n, \dots, F_1)$ be a Kingman's coalescent. Then for any fixed $(f_n, \dots, f_1) \in \mathcal{CF}_n$,

$$\mathbf{P}((F_n, \dots, F_1) = (f_n, \dots, f_1)) = \prod_{k=1}^{n-1} \mathbf{P}(F_k = f_k | (F_n, \dots, F_{k+1}) = (f_n, \dots, f_{k+1})).$$

Among the $k(k+1)$ possible oriented edges connecting roots of f_{k+1} , exactly one of them can be added to f_{k+1} to yield f_k . Thus, regardless of the sequence (f_n, \dots, f_1) ,

$$\mathbf{P}((F_n, \dots, F_1) = (f_n, \dots, f_1)) = [(n-1)!n!]^{-1}.$$

By Proposition 5.1, $T_{\mathbf{C}} \in \mathcal{RD}_n$ and it has a uniform distribution, since the bijection preserves the uniform measure of \mathbf{C} . \square

Kingman's n -coalescent is usually represented by extended binary trees with n external vertices and an increasing labeling on the $n-1$ internal vertices. The role of internal vertices is as follows. For each $k \in [n-1]$, consider the two sets of leaves in the subtrees of internal vertex labeled k ; these two sets are merged at the $(n-k)$ -th step of the coalescent. The labeling of the external vertices represent the same elements as the vertex set of t_C . The correspondence between recursive trees and increasing binary trees is mentioned, e.g., in [8, exercise II.33].

6. FURTHER RESEARCH ON THE TREE GROWTH PROCESS

The Robin-Hood pruning yields an interesting process $((T_n, \sigma_n), n \geq 1)$ which has connections to mathematical models of social and economic networks and raises challenging theoretical questions.

By Theorem 1.3 and Proposition 1.1, $\sigma(T_n) \stackrel{\text{dist}}{=} R_n$ for all $n \geq 1$. The novelty of this process is that T_n is not necessarily obtained from T_{n-1} by a simple addition of a new edge

and vertex. Rather, the Robin-Hood pruning is a fairly complex dynamic of trees. About half of the time the newly added vertex will simply attach to a uniformly random vertex, as in the standard construction of RRTs. While from time to time, a large proportion of edges will be rewired towards the newly added vertex, drastically reshaping the structure of the tree.

Note, for example, by Fact 2.3,

$$\mathbf{E}[d_{T_n}(n)] = \mathbf{E}\left[\mathbf{E}\left[\sum_{i=k}^{n-1} X_i \mid M = k\right]\right] = \sum_{k=1}^n \sum_{i=k}^{n-1} \frac{1}{n \cdot i} = \sum_{i=1}^{n-1} \sum_{k=1}^i \frac{1}{n \cdot i} = 1 - \frac{1}{n};$$

while, for any $a \in [0, 1)$,

$$\mathbf{E}[d_{T_n} \mid M \leq n^a] \geq \mathbf{E}\left[\sum_{i=n^a}^n X_i\right] = (1 - a) \ln n.$$

In the context of random networks, the Robin-Hood pruning has an interpretation in terms of ‘trends’. That is, a new vertex brings in a new idea to the network and that rewires the interests or connections of established individuals in the network. The decoration σ_n gives a ranking between the elements of T_n that determines the susceptibility of changing parents in the tree. Preferential attachment models are considered better models for real-world networks. It would be interesting to devise a similar pruning procedure that, acting on preferential attachment trees, preserves their scale-free degree distribution.

In the context of biology, Kingman’s coalescent is usually represented with increasing binary trees and, although there exists a bijection between these binary trees and n -chains, it is not clear how the Robin-Hood pruning process would have a significant interpretation in terms of the genealogical information.

Regardless of the perspective we use to motivate the process $((T_n, \sigma_n), n \geq 1)$, there are many interesting theoretical questions that would be worth pursuing. To name just a few:

- (1) Understand the process describing how the parent and descendants of a given vertex change with time.
 - Describe how the size of the subtree rooted at a fixed node j evolves.
 - How does maximum size of such subtree grow?
- (2) Understand the maximum degree in both $(R_n, n \geq 1)$ and $(T_n, n \geq 1)$.
 - How often does the maximum degree change?
 - Is this the same in both processes?

ACKNOWLEDGEMENTS

I would like to thank Louigi Addario-Berry and Henning Sulzbach for some very helpful discussions. This research was supported by FQRNT through PBEEE scholarship with number 169888.

APPENDIX A: PROOF OF PROPOSITION 3.2

The proof mimics that of [2, Proposition 4.2], but requires little more care as we wish to obtain explicit error bounds. By Proposition 5.3 we can work with the tree $T^{(n)}$ constructed from Kingman’s coalescent in Section 5.

Recall that Kingman’s coalescent consists of a chain $\mathbf{C} = (F_n, \dots, F_1)$ and that $T^{(n)}$ is the unique tree contained in F_1 . For each $v, j \in [n]$ let $T_j(v)$ denote the tree in F_j that

contains vertex v . For each $v \in [n]$, the *selection set* of v is defined as

$$\mathcal{S}_n(v) = \{2 \leq j \leq n : T_j(v) \in \{T_{a_j}^{(j)}, T_{b_j}^{(j)}\}\};$$

this set keeps record of the times when the tree containing v merges. Finally, for each $2 \leq j \leq n$, we say that ξ_j is *favourable* for vertices in $T_{a_j}^{(j)}$ (resp. vertices in $T_{b_j}^{(j)}$) if $\xi_j = 1$ (resp. $\xi_j = 0$).

The key property of Kingman's coalescent is the following. For each $j \in \mathcal{S}_n(v)$, if ξ_j favours v , then $r(T_j(v))$ increases its degree by one in the process; otherwise $r(T_j(v))$ attaches to the root of the other merging tree and the degree of $r(T_j(v))$ remains unchanged for the rest of the process. Since all vertices start the process as roots, $d_{T(n)}(v)$ is equal to the length of the first streak of favourable times for v . Moreover, $(\xi_j, j \in [n-1])$ are independent and distributed as Bernoulli $(1/2)$. Therefore we have the following distributional equivalence.

Fact 6.1. *Let D be a random variable with distribution $\text{Geo}(1/2)$ independent of $\mathcal{S}_n(v)$, then*

$$d_{T(n)}(v) \stackrel{\text{dist}}{=} \min\{D, |\mathcal{S}_n(v)|\}.$$

This fact, together with the next lemma, allow us to get estimates for the tails of $d_{T(n)}(v)$.

Lemma 6.2. *If $c \in (0, 2)$ and $0 < \varepsilon \leq 1 - c/2$. Writing $a = 1 - \varepsilon - c/2$, we have*

$$\mathbf{P}(|\mathcal{S}_n(v) \setminus [n^a]| > c \ln n) \leq O(1)n^{-\varepsilon^2/(\varepsilon+c/2)}.$$

Proof. First, there are $j(j-1)$ distinct pair of trees in F_j , exactly $j-1$ of such pairs contains $T_j(v)$; thus $\mathbf{P}(j \in \mathcal{S}_n(v)) = 2/j$. Since the merging trees are chosen independently at each time, we have that for any $a \in [0, 1)$ we have

$$|\mathcal{S}_n(v) \setminus [n^a]| \stackrel{\text{dist}}{=} \sum_{j=n^a+1}^n B_j,$$

where the variables B_1, \dots, B_n are independent Bernoulli variables with $\mathbf{E}[B_i] = 2/i$, respectively. The desired bound is then a straightforward application of Bernstein's inequalities (see, e.g. [13], Theorem 2.8 and (2.6)). For a sum S of $\{0, 1\}$ -valued variables, we have $\mathbf{P}(S \leq \mathbf{E}[S] - t) \leq \exp\{-t^2/2\mathbf{E}[S]\}$. In this case, $S = \sum_{i=n^a}^n B_i$ and

$$\mathbf{E}[S] = \sum_{i=n^a}^n \frac{2}{i} = 2(1-a) \ln n + O(1) = (c+2\varepsilon) \ln n + O(1).$$

The result follows by setting $t = 2\varepsilon \ln n + O(1)$. □

Proposition 6.3. *If $c \in (0, 2)$ and $m < c \ln n$, then for $\varepsilon = (2-c)^2/4$,*

$$2^{-m}(1 - o(n^{-\varepsilon})) \leq \mathbf{P}(d_{T(n)}(1) \geq m) \leq 2^{-m}.$$

Proof of Proposition 6.3. It follows from Lemma 6.1 that

$$\mathbf{P}(d_{T(n)}(v) \geq m) = \mathbf{P}(D \geq m) \mathbf{P}(|\mathcal{S}_n(v)| \geq m).$$

The upper bound on $\mathbf{P}(d_{T(n)}(1) \geq m)$ is then trivial, while the lower bound follows by Lemma 6.2 using $\varepsilon = 1 - c/2$ and that $\mathcal{S}_n(v) = \mathcal{S}_n(v) \setminus [1]$. □

Now, consider two distinct vertices $v, w \in [n]$. For $m \in \mathbb{N}$, let $\mathcal{G}_m \in \{2, \dots, n\}^2$ contain all pairs of selection sets that enable vertices v and w to have degree at least m ; that is, $(A, B) \in \mathcal{G}_m$ only if

$$\mathbf{P}(d_{T^{(n)}}(v) \geq m, d_{T^{(n)}}(w) \geq m, (\mathcal{S}_n(v), \mathcal{S}_n(w)) = (A, B)) > 0.$$

Since the ξ_j are independent of the selection times, we have that

$$(13) \quad \mathbf{P}(d_{T^{(n)}}(v) \geq m, d_{T^{(n)}}(w) \geq m) \geq 2^{-2m} \mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(w)) \in \mathcal{G}_m).$$

To estimate $\mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(w)) \in \mathcal{G}_m)$ we need more details on the dynamics of the model. We start with a simple tail bound for the following random variable; let

$$\tau = \max\{j : j \in \mathcal{S}_n(v) \cap \mathcal{S}_n(w)\}.$$

Lemma 6.4. *For $a \in (0, 1)$, $\mathbf{P}(\tau > n^a) \leq 4n^{-a}$.*

Proof. Vertices in $T^{(n)}$ are exchangeable, so we can take $v = 1, w = 2$; these vertices belong to distinct trees in F_j for all $j \geq \tau$. Additionally, by the ordering convention of trees in F_j , it follows that $T_j(1) = 1$ and $T_j(2) = 2$ for all $j \geq \tau$.

We claim that for all $2 < k \leq n$,

$$\mathbf{P}(\tau \leq k) = \prod_{j=k+1}^n \left(1 - \frac{2}{j(j-1)}\right).$$

This follows by induction on $n - k$. Clearly, $\tau = n$ only if $\{a_n, b_n\} = \{1, 2\}$ which occurs with probability $\frac{2}{n(n-1)}$, thus $\mathbf{P}(\tau \leq n - 1)$ satisfies the equation above. For $k < n$, we have

$$\frac{\mathbf{P}(\tau \leq k)}{\mathbf{P}(\tau \leq k+1)} = \mathbf{P}(\tau \leq k | \tau \leq k+1) = \mathbf{P}(\{a_{k+1}, b_{k+1}\} \neq \{1, 2\}) = 1 - \frac{2}{(k+1)k}.$$

Next, for k larger enough,

$$\prod_{j=k+1}^n \left(1 - \frac{2}{j(j-1)}\right) \geq \prod_{j=k}^{n-1} \left(1 - \frac{2}{j^2}\right) > 1 - \sum_{j=k}^{\infty} \frac{2}{j^2} > 1 - 4 \int_k^{\infty} x^{-2} dx = 1 - 4/k.$$

The second inequality uses that $1 - x > e^{-2x}$ for $x > 0$ sufficiently small, followed by the fact that $e^{-\sum 2x_j} > 1 - \sum 2x_j$. The result follows with $k = n^a$. \square

Lemma 6.5. *If $c \in (0, 2)$ and $m < c \ln n$, then for any $\gamma < \frac{1}{4}(1 - c + \sqrt{1 + 2c - c^2})$,*

$$\mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(w)) \in \mathcal{G}_m) \geq 1 - o(n^{-\gamma}).$$

Proof. For each $\varepsilon \in (0, 1 - c/2]$ write $a = a(\varepsilon) = 1 - \varepsilon - c/2$, then

$$(14) \quad \mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(w)) \notin \mathcal{G}_m) \leq \mathbf{P}(\tau > n^a) + 2\mathbf{P}(|\mathcal{S}_n(v) \setminus [n^a]| < c \ln n).$$

Before, establishing (14), we note that the terms in the right-hand side of (14) are bounded by Lemmas 6.4 and 6.2, respectively. Since such bounds depend on the choice of ε , we can use

$$\gamma < \max_{0 < \varepsilon \leq 1 - c/2} \left\{ \min \left(1 - \varepsilon - \frac{c}{2}, \frac{\varepsilon^2}{\varepsilon + \frac{c}{2}} \right) \right\} = \frac{1}{4} \left(1 - c + \sqrt{1 + 2c - c^2} \right).$$

The last equality since the functions to be minimized are decreasing and increasing, respectively, on the $(0, 1)$ interval. It then follows that the maximum is attained when $0 < \varepsilon < 1 - c/2$ satisfies $1 - \varepsilon - c/2 = \varepsilon^2/(\varepsilon + \frac{c}{2})$.

We now proceed to establish equation (14). At step τ , exactly one of v and w is favoured by ξ_τ . Thus, at least one of v or w gets its degree fixed for the remainder of the process. Therefore,

$$\{(\mathcal{S}_n(v), \mathcal{S}_n(w)) \in \mathcal{G}_m\} \subset \{|\mathcal{S}_n(v) \setminus [\tau]| \geq m\} \cup \{|\mathcal{S}_n(w) \setminus [\tau]| \geq m\}.$$

By intersecting with the event $\tau > n^a$, and the exchangeability of vertices in $T^{(n)}$ we get,

$$\begin{aligned} \mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(w)) \notin \mathcal{G}_m) &\leq \mathbf{P}(\tau > n^a) + 2\mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(w)) \notin \mathcal{G}_m, \tau \leq n^a) \\ &\leq \mathbf{P}(\tau > n^a) + 2\mathbf{P}(|\mathcal{S}_n(v) \setminus [\tau]| < m, \tau \leq n^a) \\ &\leq \mathbf{P}(\tau > n^a) + 2\mathbf{P}(|\mathcal{S}_n(v) \setminus [n^a]| < m, \tau \leq n^a); \end{aligned}$$

from which (14) follows. \square

Proof of Proposition 3.2. Fix $c \in (0, 2)$, $m = m(n) < c \ln n$ and let I_v, J_v be defined as in Proposition 3.2. By Proposition 5.3, it follows that $\mathbf{E}[I_v] = \mathbf{P}(d_{T^{(n)}}(v) \geq m)$ and

$$\begin{aligned} \mathbf{E}[I_v] \mathbf{E}[J_v] &= \mathbf{E}[I_v I_n] = \mathbf{P}(d_{T^{(n)}}(v) \geq m, d_{T^{(n)}}(n) \geq m) \\ &= 2^{-2m} \mathbf{P}((\mathcal{S}_n(v), \mathcal{S}_n(n)) \in \mathcal{G}_m); \end{aligned}$$

the last equality by (13). Lemmas 6.5 and 6.3 then gives that for $\alpha < \frac{1}{4}(1 - c + \sqrt{1 + 2c - c^2})$,

$$\mathbf{E}[I_v] \mathbf{E}[I_n] - \mathbf{E}[I_v] \mathbf{E}[J_{vn}] \leq 2^{-2m} - 2^{-2m}(1 + o(n^{-\alpha})) = 2^{-2m} o(n^{-\alpha}). \quad \square$$

REFERENCES

- [1] Louigi Addario-Berry. Partition functions of discrete coalescents: From cayley's formula to frieze's $\zeta(3)$ limit theorem. In *XI Symposium on Probability and Stochastic Processes: CIMAT, Mexico, November 18-22, 2013*, pages 1–45. Springer International Publishing, Cham, 2015.
- [2] Louigi Addario-Berry and Laura Eslava. High degree of random recursive trees. [arXiv:1507.05981](#), preprint (2016).
- [3] C. W. Anderson. Extreme value theory for a class of discrete distributions with applications to some stochastic processes. *J. Appl. Probability*, 7:99–113, 1970.
- [4] Luc Devroye and Jiang Lu. The strong convergence of maximal degrees in uniform random recursive trees and dags. *Random Structures Algorithms*, 7(1):1–14, 1995.
- [5] Devdatt Dubhashi and Desh Ranjan. Balls and bins: a study in negative dependence. *Random Structures Algorithms*, 13(2):99–124, 1998.
- [6] Richard Durrett. *Probability: theory and examples*. Duxbury Press, Belmont, CA, second edition, 1996.
- [7] Laura Eslava. Depth of vertices with high degree in random recursive trees. [arXiv:1611.07466](#), preprint (2016).
- [8] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
- [9] William Goh and Eric Schmutz. Limit distribution for the maximum degree of a random recursive tree. *J. Comput. Appl. Math.*, 142(1):61–82, 2002. Probabilistic methods in combinatorics and combinatorial optimization.
- [10] Christina Goldschmidt. The chen-stein method for convergence of distributions. <https://www.stats.ox.ac.uk/goldschm/chen-stein.ps.gz>, 2000.
- [11] E.J. Gumbel. Les valeurs extrêmes des distributions statistiques. *Ann. Inst. H. Poincaré*, 5(2):115–158, 1935.
- [12] Svante Janson. Asymptotic degree distribution in random recursive trees. *Random Structures Algorithms*, 26(1-2):69–83, 2005.
- [13] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. *Random graphs*. Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
- [14] Torgny Lindvall. On Strassen's theorem on stochastic domination. *Electron. Comm. Probab.*, 4:51–59 (electronic), 1999.
- [15] Malwina Łuczak and Peter Winkler. Building uniformly random subtrees. *Random Structures Algorithms*, 24(4):420–443, 2004.

- [16] Hwa Sung Na and Anatol Rapoport. Distribution of nodes of a tree by degree. *Math. Biosci.*, 6:313–329, 1970.
- [17] Jim Pitman. Coalescent random forests. *J. Comb. Theory A.*, 85:165–193, 1999.
- [18] Robert T. Smythe and Hosam M. Mahmoud. A survey of recursive trees. *Teor. ĽmovĽr. Mat. Stat.*, (51):1–29, 1994.
- [19] Thomas A. Standish. *Data Structure Techniques*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1980.

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY,
805 SHERBROOKE STREET WEST, MONTREAL, QUEBEC H3A 0B9, CANADA
E-mail address: `laura.eslavafernandez@mail.mcgill.ca`
URL: `http://www.math.mcgill.ca/eslava/`